Harry Potter and the philosophy of directed numbers

Lynn Thompson and David Bolden introduce a metaphor to discuss addition and subtraction involving directed numbers.

Addition and subtraction involving negative numbers has long been a challenge for teachers wanting to develop a conceptual understanding of these ideas in their pupils. This can lead to some teachers resorting to teaching rules such as “a negative take away a negative is a positive”. Others have searched for appropriate models which can contextualise what is going on when we want to subtract a negative number from another negative number.

Discussions with children show that they often have difficulty with this area of mathematics, perhaps because their teachers do, perhaps because they have no visual image of what is happening. Jack, presented with the calculation (-2) – (-4) stated “it’s two take aways so you add. The answer is 6”. He was asked to describe what was happening in the calculation, what pictures or models he might use to help him understand. His response was, “you can do these with money, you pretend you owe money when it’s take away signs and two take away signs mean you add. You owe £2 and you owe £4 so it’s 6”. Could Jack draw a picture to show his understanding? “No, you can’t draw what you owe, because it’s not there. You could write I owe this and I owe that next to some coins. It’s quicker to just make them adds.” This analogy of owing money, or taking away debt, when considering a negative subtracted from a negative is quite popular, but as Armstrong (2010, p3) points out, “many teachers can find this analogy confusing, rendering it both a poor explanation, and learning tool for students.”

Jack’s misconception needs to be addressed, but how can we help him make sense of what is happening here? He is using language inappropriately, “take away” instead of negative, and is applying a learned rule incorrectly. He has no visual representation to help contextualise the mathematics and is unlikely to recognise or overcome his misconception without intervention. His current understanding considers negative numbers as a quantity of a collection of objects and negative numbers do not make sense if they are considered in this way (see for example, Otten, 2009). Perhaps we need a model that will help Jack understand negative numbers using the “motion-along-a-path” metaphor (see Lakoff and Núñez, 2000).

During a subject knowledge enhancement session we held with first year undergraduate students on a teacher education course, we explored calculating with positive and negative numbers. The students knew the calculation rules and remembered the various ways they had been taught to remember which was which, but recognised that to teach effectively for understanding, they needed a model to illustrate what was happening. We began by exploring the number line model to explore how it supported children’s understanding.

Subtracting a larger number from a smaller positive number seemed fine. For example, 4 – 6 = ?. Children would know how to subtract numbers on a number line and we could model continuing left past the zero. Adding a positive number to a negative number was also clear. For example, (-3) + 7 = ? as children could locate -3 on the number line and use the number line to add beyond zero. The difficulty arose when we tried to model subtracting a negative number from another negative number, for example, (-4) – (-5) = ?. If we tried to explain the process on the number line related to the “collection of objects” model mentioned earlier we would say, “We’ve got negative 4, we take away negative 5. How many do we have now?’ Alternatively, we could draw on Lakoff and Núñez’s “motion along a path” model. In this case our question would be would be, “we are at negative 4, we subtract negative 5. Where do we end up?”

Whilst the number line may support calculation, we did not feel in this instance it fully supported understanding and so we explored other models we had seen used in the classroom. The first model used the idea of a ride in a hot air balloon where the balloons represented positive numbers and sandbags represented negative numbers. Each balloon made the basket rise one place on the number line and sandbags made the basket fall one place. Here:

- Adding balloons represents adding a positive.
- Subtracting balloons represents subtracting a positive.

- Adding sandbags represents adding a negative.
- Subtracting sandbags represents subtracting a negative.

We decided to add balloons and sandbags to a basket as necessary according to the calculation and to monitor the basket's progress on the number line. Figure 1 represents adding a positive number to a positive number, in this case 3 + 2 and figure 2 adding a positive number to a negative number.

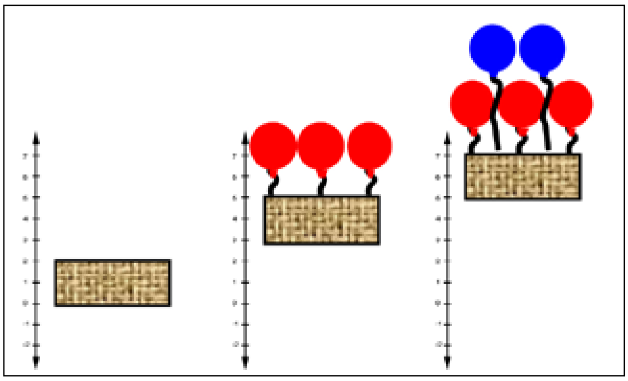

Figure 1: 3 + 2.

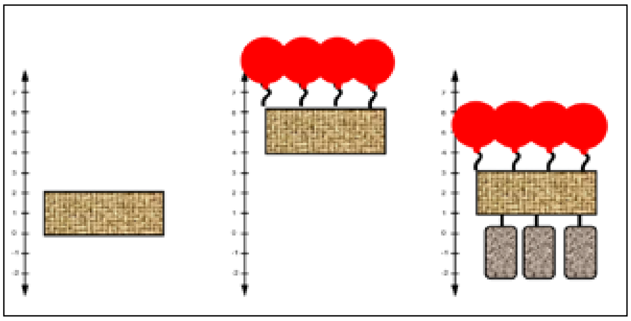

Figure 2: Adding a negative to a positive: 4 + (-3).

Students found this image useful when adding positive and negative numbers, but identified limitations when subtracting. The model worked for a calculation like (-4) – (-3). "There are 4 sandbags on the basket, it goes down to -4 on the number line. I take 3 of those sandbags away. It goes up 3 places on the number line. The basket is at -1."

The problem arose with subtractions where the number to be subtracted was greater (in representational terms) than the number to be subtracted from. For example, (-2) – (-5) = ?. Here they might say, "There are 2 sandbags on the basket. It goes down to -2 on the number line. Oh, I don't have 5 sandbags to take away." We considered adding balloons and sandbags in pairs, so as not to change the position of the basket, until we had enough sandbags to take 5 away, but felt this was becoming too complicated.

We continued to explore other models and found an article from the NCETM site very useful for examples. Gilderdale and Kiddle (2013) recognising that ''two minuses make a plus … isn't the most helpful way to think about positive and negative numbers", offered different examples to visually represent calculation with positive and negative numbers. One of their suggestions was to use hot puffs of air and sandbags, similar to our balloons and sandbags model. Other suggestions included buying and selling good and bad players for a football team and a happiness scale. Each of these models has an outcome that relates to the "motion along a path" model and we identified the same limitations in each when we came to subtracting a negative. Another model suggested by Gilderdale and Kiddle (2013) was to use counters (see figure 3) and relates to the 'quantity of a collection' model.

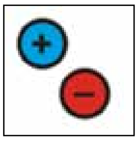

Figure 3: Positive and negative counters.

With this representation, children can see the amount of negative counters and can manipulate them to carry out a calculation. The limitations, however, are the same as with the other models. All the models we examined and which we have seen used in schools, are only able to take us so far in representing the mathematics and fall down when the negative number we want to subtract is of greater magnitude than the negative number we want to subtract from.

Our students wanted a model they could rely on to support children's understanding, and set to work in groups to develop their own ideas. We would like to share a model with you that was developed in this session, which we think is able to take us further. The context is Harry Potter and cloaks of invisibility (see figure 4):

- Positive numbers are represented by people, wizards of course.
- Negative numbers are represented by cloaks of invisibility.

Model	Calculation	Outcome
Add a wizard	Adding a positive number	We can see more wizards
Add a cloak of invisibility	Adding a negative number	We can see fewer wizards
Take away a wizard	Subtracting a positive number	We can see fewer wizards
Take away a cloak of invisibility	Subtracting a negative number	We can see more wizards

Figure 4: Harry Potter and the cloaks of invisibility.

We agreed with our students that children would be likely to relate to the Harry Potter context and would understand what the effect of wearing a cloak of invisibility would be. If children did not understand the concept, it would be a perfect excuse to read the book with them. With this model, the calculation 4 + 3 (positive plus positive) could be described as, "There are 4 wizards and 3 more wizards arrive. How many wizards can you see?" (See figure 5.)

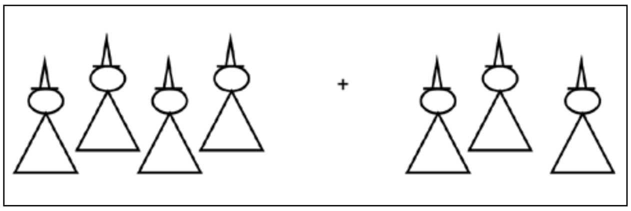

Figure 5: 4 + 3 = 7.

4 + (-3) becomes, "There are 4 wizards; we add 3 cloaks of invisibility. How many wizards can you see?"

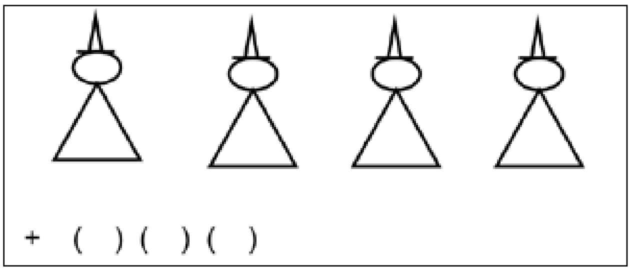

Figure 6: 4 + (-3) = 1.

And $^{-}4 + {}^{-}3$ becomes, "There are 4 cloaks of invisibility and I add 3 more cloaks of invisibility. How many cloaks of invisibility are there?"

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Figure 7: (-4) + (-3) = (-7)

Crucially $^{-}2 - {}^{-}7$, when the negative number we want to subtract is of a greater magnitude than the negative number we want to subtract from becomes, "I have 2 cloaks of invisibility and if I take away 7 cloaks of invisibility, how many wizards can I see?" Remember there are an infinite number of wizards wearing cloaks of invisibility just waiting to be revealed by having their cloaks taken away. I therefore take away the 2 cloaks I already have in my possession, cloaks taken away are represented by (   ), then I have to take cloaks from wizards until I have taken away 7 cloaks altogether. (See figure 8).

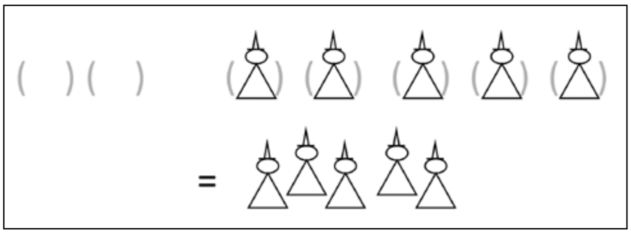

Figure 8: (-2) – (-7) = 5

With the previous representations, we had found limitations linked to subtracting negative numbers. We also noted that the models we explored were liked by some students and not by others, but all students felt that the model using cloaks of invisibility allowed discussion to take place that fostered understanding. Indeed, for some students this model allowed them to finally make sense themselves, of the rules they had learned at school. We tested our new model with Jack. We explained the concept of wizards and cloaks and modelled some simple calculations. After some sideward glances from Jack and his question "so what's this got to do with maths?'" we did make progress. At the end of our session together we asked Jack to solve (-2) – (-4). His response was "I have 2 cloaks of invisibility. I take them away and 2 more. I see 2 wizards…the answer is 2."

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